

Minimization of Transformed L_1 Penalty: Theory, Difference of Convex Function Algorithm, and Robust Application in Compressed Sensing

Shuai Zhang, and Jack Xin

Abstract

We study the minimization problem of a non-convex sparsity promoting penalty function, the transformed l_1 (TL1), and its application in compressed sensing (CS). The TL1 penalty interpolates l_0 and l_1 norms through a nonnegative parameter $a \in (0, +\infty)$, similar to l_p with $p \in (0, 1]$. TL1 is known in the statistics literature to enjoy three desired properties: unbiasedness, sparsity and Lipschitz continuity. We first consider the constrained minimization problem and prove the uniqueness of global minimizer and its equivalence to l_0 norm minimization if the sensing matrix A satisfies a restricted isometry property (RIP) and if $a > a^*$, where a^* depends only on A . Though result contains the well-known equivalence of l_1 norm and l_0 norm, in the limit $a \rightarrow +\infty$, the main difficulty is in treating the lack of scaling property of the TL1 penalty function. For a general sensing matrix A , we show that the support set of a local minimizer corresponds to linearly independent columns of A , and recall sufficient conditions for a critical point to be a local minimum. Next, we present difference of convex algorithms for TL1 (DCATL1) in computing TL1-regularized constrained and unconstrained problems in CS. The DCATL1 algorithm involves outer and inner loops of iterations, one time matrix inversion, repeated shrinkage operations and matrix-vector multiplications. For the unconstrained problem, we prove convergence of DCATL1 to a stationary point satisfying the first order optimality condition. Finally in numerical experiments, we identify the optimal value $a = 1$, and compare DCATL1 with other CS algorithms on two classes of sensing matrices: Gaussian random matrices and over-sampled discrete cosine transform matrices (ODCT). Among existing algorithms, the iterated reweighted least squares method based on $L_{1/2}$ norm is the best in sparse recovery for Gaussian matrices, and the DCA algorithm based on $L_1 - L_2$ penalty is the best for ODCT matrices. We find that for both classes of sensing matrices, the performance of DCATL1 algorithm (initiated with L_1 minimization) always ranks near the top (if not the top), and is the *most robust choice* insensitive to RIP (incoherence) of the underlying CS problems.

Keywords: Transformed l_1 penalty, sparse signal recovery theory, difference of convex function algorithm, convergence analysis, coherent random matrices, compressed sensing, robust recovery

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I. INTRODUCTION

Compressed sensing [4], [9] has generated enormous interest and research activities in mathematics, statistics, signal processing, imaging and information sciences, among numerous other areas. One of the basic problems is to reconstruct a sparse signal under a few linear measurements (linear constraints) far less than the dimension of the ambient space of the signal. Consider a sparse signal $x \in \mathbb{R}^N$, an $M \times N$ sensing matrix A and an observation $y \in \mathbb{R}^M$, $M \ll N$, such that: $y = Ax + \epsilon$, where ϵ is an N -dimensional observation error. If x is sparse enough, it can be reconstructed exactly in the noise-free case and in stable manner in the noisy case provided that the sensing matrix A satisfies certain incoherence or the restricted isometry property (RIP) [4], [9].

The direct approach is l_0 optimization, including constrained formulation:

$$\min_{x \in \mathbb{R}^N} \|x\|_0, \quad s.t. \quad y = Ax, \quad (1.1)$$

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and the unconstrained l_0 regularized optimization:

$$\min_{x \in \mathbb{R}^N} \{\|y - Ax\|_2^2 + \lambda \|x\|_0\} \quad (1.2)$$

with positive regularization parameter λ . Since minimizing l_0 norm is NP-hard [23], many viable alternatives are available. Greedy methods (matching pursuit [22], orthogonal matching pursuits (OMP) [31], and regularized OMP (ROMP) [24]) work well if the dimension N is not too large. For the unconstrained problem (1.2), the penalty decomposition method [20] replaces the term $\lambda \|x\|_0$ by $\rho_k \|x - z\|_2^2 + \lambda \|z\|_0$, and minimizes over (x, z) for a diverging sequence ρ_k . The variable z allows the iterative hard thresholding procedure.

The relaxation approach is to replace l_0 norm $\|x\|_0$ by a continuous sparsity promoting penalty functions $P(x)$. Convex relaxation uniquely selects $P(\cdot)$ as the l_1 norm $\|x\|_1$. The resulting problems are known as basis pursuit (LASSO in the over-determined regime [30]). The l_1 algorithms include l_1 -magic [4], Bregman and split Bregman methods [14], [36] and yall1 [34]. Theoretically, Candès and Tao introduced RIP condition and used it to establish the equivalent and unique global solution to l_0 minimization via l_1 relaxation among other stable recovery results [1], [2], [4].

There are also many choices of $P(\cdot)$ for non-convex relaxation. One is the l_p norm ($p \in (0, 1)$) with l_0 equivalence under RIP [7]. The $l_{1/2}$ norm is representative of this class of functions, with the reweighted least squares and half-thresholding algorithms for computation [15], [33], [32]. Near the RIP regime, $l_{1/2}$ penalty tends to have higher success rate of sparse reconstruction than l_1 . However, it is not as good as l_1 if the sensing matrix is far away from RIP [19], [35] as we shall see later as well. In the highly non-RIP (coherent) regime, it is recently found that the difference of l_1 and l_2 norm minimization gives the best sparse recovery results [35], [19]. It is therefore of both theoretical and practical interest to find a non-convex penalty that is consistently better than l_1 and always ranks among the top in sparse recovery whether the sensing matrix satisfies RIP or not.

In the statistics literature of variable selection, Fan and Li [12] advocated for classes of penalty functions with three desired properties: **unbiasedness**, **sparsity** and **continuity**. To help identify such a penalty function denoted by $\rho(\cdot)$, Fan and Lv [21] proposed the following condition for characterizing unbiasedness and sparsity promoting properties.

Condition 1. *The penalty function $\rho(\cdot)$ satisfies:*

- (i) $\rho(t)$ is increasing and concave in $t \in [0, \infty)$;
- (ii) $\rho'(t)$ is continuous with $\rho'(0+) \in (0, \infty)$;
- (iii) if $\rho(t)$ depends on a positive parameter λ , then $\rho'(t; \lambda)$ is increasing in $\lambda \in (0, \infty)$ and $\rho'(0+)$ is independent of λ .

It follows that $\rho'(t)$ is positive and decreasing, and $\rho'(0+)$ is the upper bound of $\rho'(t)$. It is shown in [12] that penalties satisfying CONDITION 1 and $\lim_{t \rightarrow \infty} \rho'(t) = 0$ enjoy both unbiasedness and sparsity. Though continuity does not generally hold for this class of penalty functions, a special one parameter family of functions, the so called **transformed l_1 functions (TL1)** $\rho_a(x)$, where

$$\rho_a(x) = \frac{(a+1)|x|}{a+|x|},$$

with $a \in (0, +\infty)$, satisfies all three desired properties [12]. We shall study the minimization of TL1 functions for CS problems, in terms of theory, algorithms and computation. We proposed the algorithms of TL1 via DC approximation [18] and implemented numerical tests based on two classes of coherent random sensing matrices. Same as $L_{1/2}$ regularization [33], [6], there also exists thresholding algorithm for TL1, which is studied in the companion paper [38].

The rest of the paper is organized as follows. In section 2, we study the properties of TL1 penalty and its regularization models. One RIP condition is given for the consistency of TL1 constrained model with original l_0 model. We also prove that the local minimizers of the TL1 constrained model extract

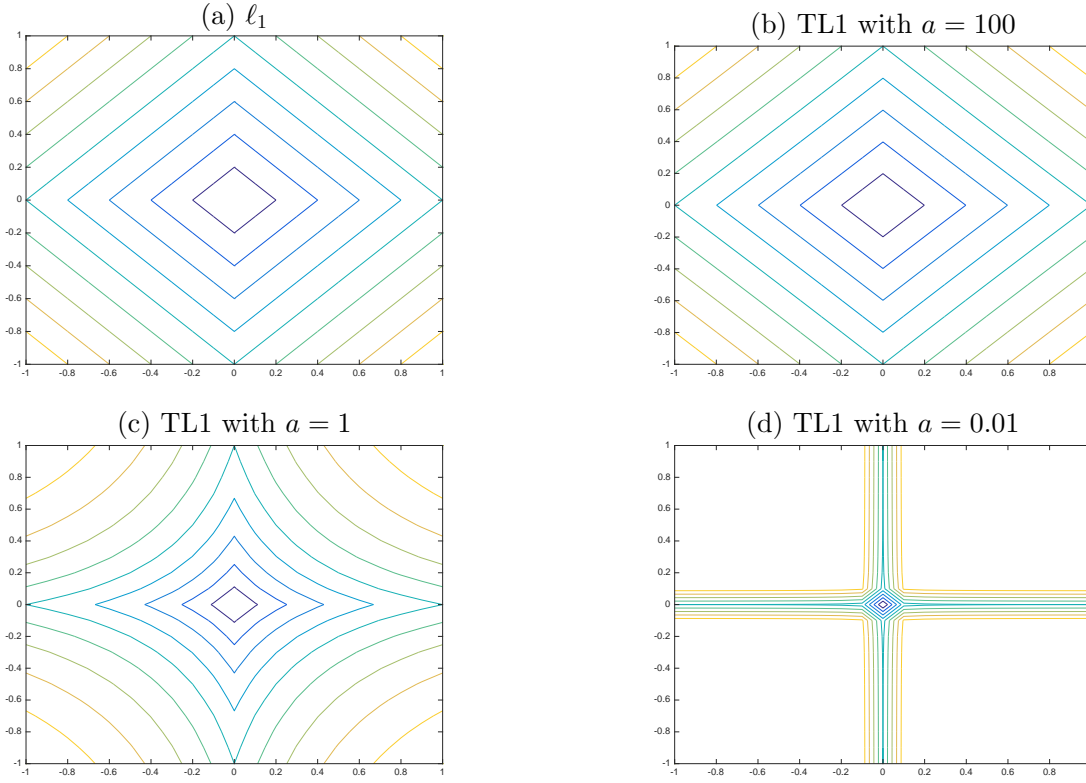


Fig. 1: Level lines of TL1 with different parameters: $a = 100$ (figure b), $a = 1$ (figure c), $a = 0.01$ (figure d). For large parameter ‘ a ’, the graph looks almost the same as l_1 (figure a). While for small value of ‘ a ’, it tends to the axis.

independent columns from the sensing matrix A , as well as the local minimizers of the unconstrained model. In section 3, we present two DC algorithms for TL1 optimization (DCATL1). In section 4, we compare the performance of DCATL1 with some state-of-the-art methods using two classes of matrices: the Gaussian and the oversampled discrete cosine transform (DCT). Numerical experiments indicate that DCATL1 is robust and consistently top ranked while maintaining high sparse recovery rates across all sensing matrices. Concluding remarks are in section 5.

II. TRANSFORMED l_1 (TL1) AND ITS REGULARIZATION MODELS

The TL1 penalty function $\rho_a(x)$ [21] is defined as

$$\rho_a(x) = \frac{(a+1)|x|}{a+|x|}, \quad (2.1)$$

where the parameter $a \in (0, +\infty)$. It interpolates the l_0 and l_1 norms as

$$\lim_{a \rightarrow 0^+} \rho_a(x) = \chi_{\{x \neq 0\}} \quad \text{and} \quad \lim_{a \rightarrow \infty} \rho_a(x) = |x|.$$

In Fig. (1), we compare level lines of l_1 and TL1 with different parameter ‘ a ’. With the adjustment of parameter ‘ a ’, the TL1 can approximate both l_1 and l_0 well. The TL1 function is Lipschitz continuous and satisfies Condition 1, thus enjoying the unbiasedness, sparsity and continuity properties [21].

Let us define TL1 regularization term $P_a(\cdot)$ as

$$P_a(x) = \sum_{i=1, \dots, N} \rho_a(x_i), \quad (2.2)$$

In the following, we consider the constrained TL1 minimization model

$$\min_{x \in \mathbb{R}^N} f(x) = \min_{x \in \mathbb{R}^N} P_a(x) \quad \text{s.t.} \quad Ax = y, \quad (2.3)$$

and the unconstrained TL1-regularized model

$$\min_{x \in \mathbb{R}^N} f(x) = \min_{x \in \mathbb{R}^N} \frac{1}{2} \|Ax - y\|_2^2 + \lambda P_a(x). \quad (2.4)$$

Some elementary inequalities of ρ_a are studied and will be used in the proof of TL1 theories.

Lemma II.1. *For $a \geq 0$, any x_i and x_j in \mathbb{R} , the following inequalities hold:*

$$\rho_a(|x_i + x_j|) \leq \rho_a(|x_i| + |x_j|) \leq \rho_a(|x_i|) + \rho_a(|x_j|) \leq 2\rho_a\left(\frac{|x_i + x_j|}{2}\right). \quad (2.5)$$

Proof: Let us prove these inequalities one by one, starting from the left.

- 1.) According to Condition 1, we know that $\rho_a(|t|)$ is increasing in the variable $|t|$. By triangle inequality $|x_i + x_j| \leq |x_i| + |x_j|$, we have:

$$\rho_a(|x_i + x_j|) \leq \rho_a(|x_i| + |x_j|).$$

2.)

$$\begin{aligned} \rho_a(|x_i|) + \rho_a(|x_j|) &= \frac{(a+1)|x_i|}{a+|x_i|} + \frac{(a+1)|x_j|}{a+|x_j|} \\ &= \frac{a(a+1)(|x_i| + |x_j| + 2|x_i x_j|/a)}{a(a+|x_i| + |x_j| + |x_i x_j|/a)} \\ &\geq \frac{(a+1)(|x_i| + |x_j| + |x_i x_j|/a)}{(a+|x_i| + |x_j| + |x_i x_j|/a)} \\ &= \rho_a(|x_i| + |x_j| + |x_i x_j|/a) \\ &\geq \rho_a(|x_i| + |x_j|). \end{aligned}$$

- 3.) By concavity of the function $\rho_a(\cdot)$,

$$\frac{\rho_a(|x_i|) + \rho_a(|x_j|)}{2} \leq \rho_a\left(\frac{|x_i| + |x_j|}{2}\right).$$

■

Remark II.1. *It follows from Lemma II.1 that the triangular inequality holds for the function $\rho(x) \equiv \rho_a(|x|)$: $\rho(x_i + x_j) = \rho_a(|x_i + x_j|) \leq \rho_a(|x_i|) + \rho_a(|x_j|) = \rho(x_i) + \rho(x_j)$. Also we have: $\rho(x) \geq 0$, and $\rho(x) = 0 \Leftrightarrow x = 0$. Our penalty function ρ acts almost like a norm. However, it lacks absolute scalability, or $\rho(cx) \neq |c|\rho(x)$ in general. The next lemma further analyses this inequality.*

Lemma II.2.

$$\rho_a(|cx|) = \begin{cases} \leq |c|\rho_a(|x|) & \text{if } |c| > 1; \\ \geq |c|\rho_a(|x|) & \text{if } |c| \leq 1. \end{cases} \quad (2.6)$$

Proof:

$$\begin{aligned} \rho_a(|cx|) &= \frac{(a+1)|c||x|}{a+|c||x|} \\ &= |c|\rho_a(|x|) \frac{a+|x|}{a+|cx|}. \end{aligned}$$

So if $|c| \leq 1$, the factor $\frac{a+|x|}{a+|cx|} \geq 1$. Then $\rho_a(|cx|) \geq |c|\rho_a(|x|)$. Similarly when $|c| > 1$, we have $\rho_a(|cx|) \leq |c|\rho_a(|x|)$. ■

A. RIP Condition for Constrained Model

For the constrained TL1 model (2.3), we present a theory on sparse recovery based on RIP [2]. Suppose β^0 is a sparsest solution for l_0 minimization s.t. $A\beta^0 = y$, while another vector β is defined as

$$\beta = \arg \min_{\beta \in \mathbb{R}_N} \{P_a(\beta) \mid A\beta = y\}. \quad (2.7)$$

We addressed the question whether the two vectors β and β^0 are equal to each other. That is to say, under what condition we can recover the sparsest solution β^0 via solving the relaxation problem (2.3).

For an $M \times N$ matrix A and set $T \subset \{1, \dots, N\}$, let A_T be the matrix consisting of the column a_j of A for $j \in T$. Similarly for vector x , x_T is a sub-vector, consisting of components indexed from the set T .

Definition II.1. (*Restricted Isometry Constant*) For each number s , define the s -restricted isometry constant of matrix A as the smallest number δ_s such that for all subset T with $|T| \leq s$ and all $x \in \mathbb{R}_{|T|}$, the inequality

$$(1 - \delta_s)\|x\|_2^2 \leq \|A_T x\|_2^2 \leq (1 + \delta_s)\|x\|_2^2$$

holds.

For a fixed y , the under-determined linear system has infinitely many solutions. Let x be one solution of $Ax = y$. It does not need to be the l_0 or ρ_a minimizer. If $P_a(x) > 1$, we scale y by the positive scalar C as:

$$y_C = \frac{y}{C}; \quad x_C = \frac{x}{C}. \quad (2.8)$$

Now x_C is a solution to the modified problem: $Ax_C = y_C$. When C becomes larger, the number $P_a(x_C)$ is smaller and tends to 0 in the limit $C \rightarrow \infty$. Thus, we can find a constant $C \geq 1$, such that $P_a(x_C) \leq 1$. That is to say, for scaled vector x_C , we always have: $P_a(x_C) \leq 1$.

Since the penalty $\rho_a(t)$ is increasing in positive variable t , we have the inequality:

$$\begin{aligned} P_a(x_C) &\leq |T|\rho_a(|x_C|_\infty) \\ &= |T|\rho_a\left(\frac{|x|_\infty}{C}\right) \\ &= \frac{|T|(a+1)|x|_\infty}{aC + |x|_\infty}, \end{aligned}$$

where $|T|$ is the cardinality of the support set of vector x . For $P_a(x_C) \leq 1$, it suffices to impose:

$$\frac{|T|(a+1)|x|_\infty}{aC + |x|_\infty} \leq 1,$$

or:

$$C \geq \frac{|x|_\infty}{a} (a|T| + |T| - 1). \quad (2.9)$$

Let β^0 be the l_0 minimizer for the constrained l_0 optimization problem (1.1) with support set T . Due to the scale-invariance of l_0 , β_C^0 (defined similarly as above) is a global l_0 minimizer for the modified problem:

$$\min \|\beta\|_0, \quad \text{s.t.} \quad y_C = A\beta. \quad (2.10)$$

with the same support set T .

Then for the modified ρ_a optimization:

$$\min P_a(\beta), \quad \text{s.t.} \quad y_C = A\beta, \quad (2.11)$$

we have the following RIP condition.

Theorem II.1. (*TL1 Exact Sparse Recovery*)

For a given sensing matrix A , if there is a number $R > |T|$, such that

$$\delta_R + \frac{R}{|T|} \delta_{R+|T|} < \frac{R}{|T|} - 1, \quad (2.12)$$

then there exists $a^* > 0$, depending only on matrix A , such that for any $a > a^*$, the minimizer β_C for (2.11) is unique and equal to the minimizer β_C^0 in (2.10) for any C satisfying (2.9).

The proof is in the appendix.

Remark II.2. Theorem II.1 contains a sufficient condition for β to be the unique global minimizer of l_0 optimization problem (1.1). On the other hand, with a choice of $R = 3|T|$, our condition (2.12) becomes:

$$\delta_{3|T|} + 3\delta_{4|T|} < 2, \quad (2.13)$$

which is exactly the condition (1.6) of Theorem 1.1 in [2]. This is consistent with the fact that when parameter a goes to $+\infty$, our penalty function ρ_a recovers the l_1 norm.

Next, we prove that TL1 recovery is stable under noisy measurements, i.e.,

$$\min P_a(\beta), \quad \text{s.t.} \quad \|y_C - A\beta\|_2 \leq \tau. \quad (2.14)$$

Theorem II.2. (Stable Recovery Theory)

Under the same RIP condition and a^* in theorem II.1, for $a > a^*$, the solution β_C^n for optimization (2.14) satisfies

$$\|\beta_C^n - \beta_C^0\|_2 \leq D\tau,$$

for some constant D depending only on the RIP condition.

Proof: Set $n = A\beta - y_C$. In the proof, we use three related notations listed below for clarity:

- (i) $\beta_C^n \Rightarrow$ optimal solution for the noisy constrained problem (2.14);
- (ii) $\beta_C \Rightarrow$ optimal solution for the noiseless constrained problem (2.11);
- (iii) $\beta_C^0 \Rightarrow$ optimal solution for the l_0 problem (2.10).

Let T be the support set of β_C^0 , i.e., $T = \text{supp}(\beta_C^0)$, and vector $e = \beta_C^n - \beta_C^0$. Following the proof of Theorem II.1, we obtain:

$$\sum_{j=2}^L \|e_{T_j}\|_2 \leq \sum_{j=1}^L \frac{P_a(e_{T_j})}{R^{1/2}} = \frac{P_a(e_{T^c})}{R^{1/2}}$$

and

$$\|e_{T_{01}}\|_2 \geq \frac{a}{(a+1)\sqrt{|T|}} P_a(e_T).$$

Further, due to the inequality $P_a(\beta_{T^c}^n) = P_a(e_{T^c}) \leq P_a(e_T)$ and inequality (1.4), we get

$$\|Ae\|_2 \geq \frac{P_a(e_T)}{R^{1/2}} C_\delta,$$

where $C_\delta = \sqrt{1 - \delta_{R+|T|}} \frac{a}{a+1} \sqrt{\frac{R}{|T|}} - \sqrt{1 + \delta_R}$.

By the initial assumption on the size of observation noise, we have

$$\|Ae\|_2 = \|A\beta_C^n - A\beta_C^0\|_2 = \|n\|_2 \leq \tau, \quad (2.15)$$

so we have: $P_a(e_T) \leq \frac{\tau R^{1/2}}{C_\delta}$.

On the other hand, we know that $P_a(\beta_C) \leq 1$ and β_C is in the feasible set of the noisy problem (2.14). Thus we have the inequality: $P_a(\beta_C^n) \leq P_a(\beta_C) \leq 1$. By (1.6), $\beta_{C,i}^n \leq 1$ for each i . So, we have

$$|\beta_{C,i}^n| \leq \rho_a(|\beta_{C,i}^n|). \quad (2.16)$$

It follows that

$$\begin{aligned} \|e\|_2 &\leq \|e_T\|_2 + \|e_{T^c}\|_2 = \|e_T\|_2 + \|\beta_{C,T^c}^n\|_2 \\ &\leq \frac{\|A_T e_T\|_2}{\sqrt{1-\delta_T}} + \|\beta_{C,T^c}^n\|_1 \\ &\leq \frac{\|A_T e_T\|_2}{\sqrt{1-\delta_T}} + P_a(\beta_{C,T^c}^n) = \frac{\|A_T e_T\|_2}{\sqrt{1-\delta_T}} + P_a(e_{T^c}) \\ &\leq \frac{\tau}{\sqrt{1-\delta_R}} + P_a(e_T) \leq D\tau. \end{aligned}$$

where constant number D depends on δ_R and $\delta_{R+|T|}$. The second inequality uses the definition of RIP, while the first inequality in the last row comes from (2.15). ■

B. Sparsity of Local Minimizer

We study properties of local minimizers of both the constrained problem (2.3) and the unconstrained model (2.4). As in l_p and l_{1-2} minimization [35], [19], a local minimizer of TL1 minimization extracts linearly independent columns from the sensing matrix A , with no requirement for A to satisfy RIP. Reversely, we state additional conditions on A for a stationary point to be a local minimizer besides the linear independence of the corresponding column vectors.

Theorem II.3. (Local minimizer of constrained model)

Suppose x^* is a local minimizer of the constrained problem (2.3) and $T^* = \text{supp}(x^*)$, then A_{T^*} is of full column rank, i.e. columns of A_{T^*} are linearly independent.

Proof: Here we argue by contradiction. Suppose that the column vectors of A_{T^*} are not linearly independent, then there exists non-zero vector $v \in \ker(A)$, such that $\text{supp}(v) \subseteq T^*$. For any neighbourhood of x^* , $N(x^*, r)$, we can scale v so that:

$$\|v\|_2 \leq \min\{r; |x_i^*|, i \in T^*\}. \quad (2.17)$$

Next we define:

$$\begin{aligned} \xi_1 &= x^* + v; \\ \xi_2 &= x^* - v, \end{aligned}$$

so both ξ_1 and ξ_2 , $\in \mathcal{B}(x^*, r)$, and $x^* = \frac{1}{2}(\xi_1 + \xi_2)$. On the other hand, from $\text{supp}(v) \subseteq T^*$, we have that $\text{supp}(\xi_1), \text{supp}(\xi_2) \subseteq T^*$. Moreover, due to the inequality (2.17), vectors x^* , ξ_1 , and ξ_2 are located in the same orthant, i.e. $\text{sign}(x_i^*) = \text{sign}(\xi_{1,i}) = \text{sign}(\xi_{2,i})$, for any index i . It means that $\frac{1}{2}|\xi_1| + \frac{1}{2}|\xi_2| = \frac{1}{2}|\xi_1 + \xi_2|$. Since the penalty function $P_a(t)$ is strictly concave for non-negative variable t ,

$$\begin{aligned} \frac{1}{2}P_a(\xi_1) + \frac{1}{2}P_a(\xi_2) &= \frac{1}{2}P_a(|\xi_1|) + \frac{1}{2}P_a(|\xi_2|) \\ &< P_a(\frac{1}{2}|\xi_1| + \frac{1}{2}|\xi_2|) = P_a(\frac{1}{2}|\xi_1 + \xi_2|) = P_a(x^*). \end{aligned}$$

So for any fixed r , we can find two vectors ξ_1 and ξ_2 in the neighbourhood $\mathcal{B}(x^*, r)$, such that $\min\{P_a(\xi_1), P_a(\xi_2)\} \leq \frac{1}{2}P_a(\xi_1) + \frac{1}{2}P_a(\xi_2) < P_a(x^*)$. Both vectors are in the feasible set of the constrained problem (2.3), in contradiction with the assumption that x^* is a local minimizer. ■

The same property also holds for the local minimizers of unconstrained model (2.4), because a local minimizer of the unconstrained problem is also a local minimizer for a constrained optimization model [2], [35]. We skip the details and state the result below.

Theorem II.4. (Local minimizer of unconstrained model)

Suppose x^* is a local minimizer of the unconstrained problem (2.4) and $T^* = \text{supp}(x^*)$, then columns of A_{T^*} are linearly independent.

Remark II.3. From the two theorems above, we conclude the following facts:

- (i) For any local minimizer of (2.3) or (2.4), e.g. x^* , the sparsity of x^* is at most $\text{rank}(A)$;
- (ii) The number of local minimizers is finite, for both problem (2.3) and (2.4).

In [21], the authors studied sufficient conditions of a strict local minimizer for minimizing any penalty functions satisfying Condition 1. Here we specialize and simplify it for our concave TL1 function ρ_a .

For a convex function $h(\cdot)$, the subdifferential $\partial h(x)$ at $x \in \text{dom } h$ is the closed convex set:

$$\partial h(x) := \{y \in \mathbb{R}^N : h(z) \geq h(x) + \langle z - x, y \rangle, \quad \forall z \in \mathbb{R}^N\}, \quad (2.18)$$

which generalizes the derivative in the sense that h is differentiable at x if and only if $\partial h(x)$ is a singleton or $\{\nabla h(x)\}$.

The TL1 penalty function $p_a(\cdot)$ can be written as a difference of two convex functions:

$$\begin{aligned} \rho_a(t) &= \frac{(a+1)|t|}{a+|t|} \\ &= \frac{(a+1)|t|}{a} - \left(\frac{(a+1)|t|}{a} - \frac{(a+1)|t|}{a+|t|} \right) \\ &= \frac{(a+1)|t|}{a} - \frac{(a+1)t^2}{a(a+|t|)}. \end{aligned} \quad (2.19)$$

Thus we can define the general derivative of function $P_a(\cdot)$, as the difference of two convex derivatives,

$$\partial P_a(x) = \frac{a+1}{a} \partial \|x\|_1 - \partial \varphi_a(x), \quad (2.20)$$

where $\partial \|x\|_1$ is the subdifferential of $\|x\|_1$ and

$$\varphi_a(x) = \frac{a+1}{a} \|x\|_1 - P_a(x) = \sum_{i=1}^N \frac{(a+1)|x_i|^2}{a(a+|x_i|)}, \quad (2.21)$$

which is differentiable. As we know, $\partial \|x\|_1 = \{\text{sgn}(x_i)\}_{i=1,\dots,N}$, where

$$\text{sgn}(t) = \begin{cases} \text{sign}(t), & \text{if } t \neq 0, \\ [-1, 1], & \text{otherwise.} \end{cases} \quad (2.22)$$

Definition II.2. (Maximum concavity and local concavity of the penalty function)

For a penalty function ρ , we define its maximum concavity as:

$$\kappa(\rho) = \sup_{t_1, t_2 \in (0, \infty), t_1 < t_2} - \frac{\rho'(t_2) - \rho'(t_1)}{t_2 - t_1} \quad (2.23)$$

and its local concavity of ρ at a point $b = (b_1, b_2, \dots, b_R)^t \in \mathbb{R}^R$ with $\|b\|_0 = R$ as:

$$\kappa(\rho; b) = \lim_{\epsilon \rightarrow 0+} \max_{1 \leq j \leq R} \sup_{t_1, t_2 \in (|b_j|-\epsilon, |b_j|+\epsilon), t_1 < t_2} - \frac{\rho'(t_2) - \rho'(t_1)}{t_2 - t_1}. \quad (2.24)$$

In [21], Lv and Fan proposed a set of sufficient conditions for the (strict) local minimizer of (2.4).

Condition 2. For vector $\beta \in \mathbb{R}^N$, $\lambda > 0$ with the support set $T = \text{supp}(\beta)$:

- (i) Matrix $Q = A_T^t A_T$ is non-singular, i.e. matrix A_T is column independent;
- (ii) For vector $z = \frac{1}{\lambda} A^t(y - A\beta)$, $\|z_{T^c}\|_\infty < \rho'_a(0+) = \frac{a+1}{a}$;

- (iii) Vector β_T satisfies the stationary point equation: $\beta_T = Q^{-1}A_T^t y - \lambda Q^{-1}w_T$, where $w_T \in \partial P_a(\beta)$;
- (iv) $\lambda_{\min}(Q) \geq \lambda \kappa(\rho_a; \beta_T)$, where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalues of a given symmetric matrix.

Here we present the theory and give a simplified proof to illustrate (i)-(iv) in Condition 2.

Theorem II.5. *If a vector $\beta \in \mathbb{R}^N$ satisfies all four requirements in Condition 2, then β is a local minimizer of problem (2.4).*

Furthermore, if the inequality of (iv) in Condition 2 is strict, then the vector β is a strict local minimizer.

Proof: Let us define a subspace of \mathbb{R}^N as: $S = \{\beta \in \mathbb{R}^N | \beta_{T^c} = 0\}$ and denote the optimal objective function as

$$\ell(x) = 2^{-1} \|Ax - y\|_2^2 + \lambda P_a(x) = 2^{-1} \|Ax - y\|_2^2 + \lambda \sum_{j=1}^N \rho_a(x_j).$$

First, by Condition 1, (iv) of Condition 2, and the definition of $\kappa(\rho_a; \beta_T)$, the objective function $\ell(\cdot)$ is convex in ball area $\mathcal{B}(\beta, r_0) \cap S$, where r_0 is a positive number (the radius). By equation (iii) in Condition 2, β_T is a local minimizer of $\ell(\cdot)$ in S .

Next, we show that the sparse vector β is indeed a local minimizer of $\ell(x)$ in \mathbb{R}^N . Because of the inequality (ii) in Condition 2, there exists a $\delta \in (0, \infty)$ and a positive number $r_1 < \delta$, such that

$$\|w(x)_{T^c}\|_\infty < \rho'_a(\delta) \leq \rho'_a(0+),$$

for any vector $x \in \mathcal{B}(\beta, r_1)$ and $w(x) = \frac{1}{\lambda} A^t(y - Ax)$. We can further shrink r_1 if necessary so that $r_1 < r_0$, and then $\mathcal{B}(\beta, r_1) \subseteq \mathcal{B}(\beta, r_0)$.

$\forall \beta_1 \in \mathcal{B}(\beta, r_1)$, and define β_2 as the projection of β_1 onto set S . Then each related element pairs from β_1 and β_2 sits at the same side of 1-dimensional x-axis, where $\partial P_a(x)$ is well defined for x lying between β_1 and β_2 . Thus we have

$$\ell(\beta_1) = \ell(\beta_2) + \nabla^t \ell(\beta_0)(\beta_1 - \beta_2),$$

where β_0 lies on the line segment joining β_1 and β_2 .

Furthermore, due to these facts

$$(\beta_1 - \beta_2)_T = 0, \quad \beta_0 \in \mathcal{B}(\beta, r_1) \quad \text{and} \quad \text{sign}(\beta_{0,T^c}) = \text{sign}(\beta_{1,T^c}),$$

we have the following inequality:

$$\begin{aligned} \ell(\beta_1) - \ell(\beta_2) &= \partial \ell(\beta_0)_{T^c} * \beta_{1,T^c} \\ &= -\lambda [\lambda^{-1} A_{T^c}^t (y - A\beta_0)]^t \beta_{1,T^c} + \lambda \sum_{j \in T^c} \rho'_a(\beta_{0,j}) \beta_{1,j} \\ &> -\lambda \rho'_a(\delta) \|\beta_{1,T^c}\|_1 + \lambda \sum_{j \in T^c} \rho'_a(|\beta_{0,j}|) |\beta_{1,j}| \\ &\geq -\lambda \rho'_a(\delta) \|\beta_{1,T^c}\|_1 + \lambda \rho'_a(\delta) \|\beta_{1,T^c}\|_1 = 0, \end{aligned}$$

where $*$ stands for vector cross product and we also used the fact for $j \in T^c$, $|\beta_{0,j}| \leq \delta$.

So for any $\beta_1 \in \mathcal{B}(\beta, r_1)$, $\ell(\beta_1) > \ell(\beta_2)$. Since β_2 is a projection on S and it belongs to the ball $\mathcal{B}(\beta, r_1) \subseteq \mathcal{B}(\beta, r_0)$,

$$\ell(\beta_1) > \ell(\beta_2) \geq \ell(\beta).$$

The (iv) in Condition 2 is only used in the first part of the proof. If we has the strict inequality $\lambda_{\min}(Q) > \lambda \kappa(\rho_a; \beta_T)$, then β_T is a strict local minimizer in S , as the function $\ell(\cdot)$ is strictly convex in the intersection $\mathcal{B}(\beta, r_0) \cap S$. Further, the same proof shows that β is a strict local minimizer in \mathbb{R}^N . ■

III. DC ALGORITHM FOR TRANSFORMED l_1 PENALTY

DC (Difference of Convex functions) Programming and DCA (DC Algorithms) was introduced in 1985 by Pham Dinh Tao, and extensively developed by Le Thi Hoai An and Pham Dinh Tao to become a useful tool for non-convex optimization ([26], [16] and references therein).

A standard DC program is of the form

$$\alpha = \inf\{f(x) = g(x) - h(x) : x \in \mathbb{R}^n\} \quad (P_{dc}),$$

where g, h are lower semicontinuous proper convex functions on \mathbb{R}^n . Here f is called a DC function, while $g - h$ is a DC decomposition of f .

The DCA is an iterative method and generates a sequence $\{x^k\}$. For example, at the current point x^l of iteration, function $h(x)$ is approximated by its affine minorization $h_l(x)$, defined by

$$h_l(x) = h(x^l) + \langle x - x^l, y^l \rangle, \quad y^l \in \partial h(x^l).$$

Then the original model is converted to solve a convex program in the form:

$$\inf\{g(x) - h_l(x) : x \in \mathbb{R}^d\} \Leftrightarrow \inf\{g(x) - \langle x, y^l \rangle : x \in \mathbb{R}^d\},$$

where the optimal solution is denoted as x^{l+1} .

A. Algorithm for Unconstrained Model — DCATLI

For the following unconstrained optimization problem (2.4):

$$\min_{x \in \mathbb{R}^N} f(x) = \min_{x \in \mathbb{R}^N} \frac{1}{2} \|Ax - y\|_2^2 + \lambda P_a(x),$$

we propose a DC decomposition scheme $f(x) = g(x) - h(x)$, where

$$\begin{cases} g(x) &= \frac{1}{2} \|Ax - y\|_2^2 + c\|x\|_2^2 + \lambda \frac{(a+1)}{a} \|x\|_1; \\ h(x) &= \lambda \varphi_a(x) + c\|x\|_2^2. \end{cases} \quad (3.1)$$

Here function $\varphi_a(x)$ is defined in equation (2.21). Thus function $h(x)$ is differentiable. Additional factor $c\|x\|_2^2$ with hyperparameter c is used to improve the convexity of these two functions, and will be used in the convergence theorem.

Algorithm 1: DCA for unconstrained transformed l_1 penalty minimization

Define: $\epsilon_{outer} > 0$

Initialize: $x^0 = 0, n = 0$

while $|x^{n+1} - x^n| > \epsilon_{outer}$ **do**

$v^n = \partial h(x^n) = \partial \varphi_a(x^n) + 2cx^n$

$x^{n+1} = \arg \min_{x \in \mathbb{R}^N} \{ \frac{1}{2} \|Ax - y\|_2^2 + c\|x\|_2^2 + \lambda \frac{(a+1)}{a} \|x\|_1 - \langle x, v^n \rangle \}$

then $n + 1 \rightarrow n$

end while

At each step, we need to solve a strongly convex l_1 -regularized sub-problem, which is:

$$\begin{aligned} x^{n+1} &= \arg \min_{x \in \mathbb{R}^N} \{ \frac{1}{2} \|Ax - y\|_2^2 + c\|x\|_2^2 + \lambda \frac{(a+1)}{a} \|x\|_1 - \langle x, v^n \rangle \} \\ &= \arg \min_{x \in \mathbb{R}^N} \{ \frac{1}{2} x^t (A^t A + 2cI) x - \langle x, v^n + A^t y \rangle + \lambda \frac{(a+1)}{a} \|x\|_1 \}. \end{aligned} \quad (3.2)$$

We now employ the Alternating Direction Method of Multipliers (ADMM). The sub-problem is recast as:

$$\begin{aligned} x^{n+1} &= \arg \min_{x \in \mathbb{R}^N} \{ \frac{1}{2} x^t (A^t A + 2cI) x - \langle x, v^n + A^t y \rangle + \lambda \frac{(a+1)}{a} \|z\|_1 \} \\ &\quad s.t. \quad x - z = 0. \end{aligned} \quad (3.3)$$

Define the augmented Lagrangian function as:

$$L(x, z, u) = \frac{1}{2}x^t(A^tA + 2cI)x - \langle x, v^n + A^ty \rangle + \lambda \frac{(a+1)}{a} \|z\|_1 + \frac{\delta}{2} \|x - z\|_2^2 + u^t(x - z),$$

where u is the Lagrange multiplier, and $\delta > 0$ is a penalty parameter. The ADMM consists of three iterations:

$$\begin{cases} x^{k+1} = \arg \min_x L(x, z^k, u^k); \\ z^{k+1} = \arg \min_z L(x^{k+1}, z, u^k); \\ u^{k+1} = u^k + \delta(x^{k+1} - z^{k+1}). \end{cases}$$

The first two steps have closed-form solutions and are described in Algorithm 2, where $shrink(.,.)$ is a soft-thresholding operator given by:

$$shrink(x, r)_i = \text{sgn}(x_i) \max\{|x_i| - r, 0\}.$$

Algorithm 2: ADMM for subproblem (3.2)

Initial guess: x^0, z^0, u^0 and iterative index $k = 0$
while not converged **do**
 $x^{k+1} := (A^tA + 2cI + \delta I)^{-1}(A^ty - v^n + \delta z^k - u^k)$
 $z^{k+1} := shrink(x^{k+1} + u^k, \frac{a+1}{a\delta}\lambda)$
 $u^{k+1} := u^k + \delta(x^{k+1} - z^{k+1})$
 then $k + 1 \rightarrow k$
end while

B. Convergence Theory for Unconstrained DCATL1

We present a convergence theory for the Algorithm 1 (DCATL1). We prove that the sequence $\{f(x^n)\}$ is decreasing and convergent, while the sequence $\{x^n\}$ is bounded under some requirement on λ . Its sub-limit vector x^* is a stationary point satisfying the first order optimality condition. Our proof is based on the convergent theory of DCA for $l_1 - l_2$ penalty function [35] besides the general results [27], [28].

Definition III.1. (Modulus of strong convexity) For a convex function $f(x)$, the modulus of strong convexity of f on \mathbb{R}^N , denoted as $m(f)$, is defined by

$$m(f) := \sup\{\rho > 0 : f - \frac{\rho}{2} \|\cdot\|_2^2 \text{ is convex on } \mathbb{R}^N\}.$$

Let us recall a useful inequality from Proposition A.1 in [28] concerning the sequence $f(x^n)$.

Lemma III.1. Suppose that $f(x) = g(x) - h(x)$ is a D.C. decomposition, and the sequence $\{x^n\}$ is generated by (3.2), then

$$f(x^n) - f(x^{n+1}) \geq \frac{m(g) + m(h)}{2} \|x^{n+1} - x^n\|_2^2.$$

Here is the convergence theory for our unconstrained Algorithm 1 — DCATL1. The objective function is : $f(x) = \frac{1}{2}\|Ax - y\|_2^2 + \lambda P_a(x)$.

Theorem III.1. The sequences $\{x^n\}$ and $\{f(x^n)\}$ in Algorithm 1 satisfy:

- 1) Sequence $\{f(x^n)\}$ is decreasing and convergent.
- 2) $\|x^{n+1} - x^n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. If $\lambda > \frac{\|y\|_2^2}{2(a+1)}$, $\{x^n\}_{n=1}^\infty$ is bounded.
- 3) Any subsequential limit vector x^* of $\{x^n\}$ satisfies the first order optimality condition:

$$0 \in A^T(Ax^* - y) + \lambda \partial P_a(x^*), \quad (3.4)$$

implying that x^* is a stationary point of (2.4).

Proof:

1) By the definition of $g(x)$ and $h(x)$ in equation (3.1), it is easy to see that:

$$\begin{aligned} m(g) &\geq 2c; \\ m(h) &\geq 2c. \end{aligned}$$

By Lemma III.1, we have:

$$\begin{aligned} f(x^n) - f(x^{n+1}) &\geq \frac{m(g) + m(h)}{2} \|x^{n+1} - x^n\|_2^2 \\ &\geq 2c \|x^{n+1} - x^n\|_2^2. \end{aligned}$$

So the sequence $\{f(x^n)\}$ is decreasing and non-negative, thus convergent.

2) It follows from the convergence of $\{f(x^n)\}$ that:

$$\|x^{n+1} - x^n\|_2^2 \leq \frac{f(x^n) - f(x^{n+1})}{2c} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

If $y = 0$, since the initial vector $x^0 = 0$, and the sequence $\{f(x^n)\}$ is decreasing, we have $f(x^n) = 0$, $\forall n \geq 1$. So $x^n = 0$, and the boundedness holds.

Consider non-zero vector y . Then

$$f(x^n) = \frac{1}{2} \|Ax^n - y\|_2^2 + \lambda P_a(x^n) \leq f(x^0) = \frac{1}{2} \|y\|_2^2,$$

So $\lambda P_a(x^n) \leq \frac{1}{2} \|y\|_2^2$, implying $2\lambda\varphi_a(\|x^n\|_\infty) \leq \|y\|_2^2$, or:

$$\frac{2\lambda(a+1)\|x^n\|_\infty}{a + \|x^n\|_\infty} \leq \|y\|_2^2.$$

So if $\lambda > \frac{\|y\|_2^2}{2(a+1)}$, then

$$\|x^n\|_\infty \leq \frac{a\|y\|_2^2}{2\lambda(a+1) - \|y\|_2^2}.$$

Thus the sequence $\{x^n\}_{n=1}^\infty$ is bounded.

3) Let $\{x^{n_k}\}$ be a subsequence of $\{x^n\}$ which converges to x^* . So the optimality condition at the n_k -th step of Algorithm 1 is expressed as:

$$\begin{aligned} 0 \in & A^T(Ax^{n_k} - y) + 2c(x^{n_k} - x^{n_k-1}) \\ & + \lambda\left(\frac{a+1}{a}\right)\partial\|x^{n_k}\|_1 - \lambda\partial\varphi_a(x^{n_k-1}). \end{aligned} \quad (3.5)$$

Since $\|x^{n+1} - x^n\|_2 \rightarrow 0$ as $n \rightarrow \infty$ and x^{n_k} converges to x^* , as shown in Proposition 3.1 of [35], we have that for sufficiently large index n_k ,

$$\partial\|x^{n_k}\|_1 \subseteq \partial\|x^*\|_1.$$

Letting $n_k \rightarrow \infty$ in (3.5), we have

$$0 \in A^T(Ax^* - y) + \lambda\left(\frac{a+1}{a}\right)\partial\|x^*\|_1 - \lambda\partial\varphi_a(x^*).$$

By the definition of $\partial P_a(x)$ at (2.20), we have $0 \in A^T(Ax^* - y) + \lambda\partial P_a(x^*)$.

■

Remark III.1. The above theorem says that the sub-sequence limit x^* is a stationary point for (2.4). Let $T^* = \text{supp}(x^*)$, there exists vector $w \in \partial P_a(x^*)$, s.t.

$$\begin{aligned} 0 &= A^t(Ax^* - y) + \lambda w \\ \Rightarrow 0 &= A_{T^*}^t(A_{T^*}x_{T^*}^* - y) + \lambda w_{T^*} \\ \Rightarrow 0 &= Qx_{T^*}^* - A_{T^*}^t y + \lambda w_{T^*} \\ \Rightarrow x_{T^*}^* &= Q^{-1}A_{T^*}^t y - \lambda Q^{-1}w_{T^*}. \end{aligned} \quad (3.6)$$

So (iii) of Condition 2 is automatically satisfied by x^* . If (i), (ii) and (iv) are also satisfied, the limit point x^* is a local minimizer of (2.4).

C. Algorithm for Constrained Model

Here we also give a DCA scheme to solve the constrained problem (2.3)

$$\begin{aligned} \min_{x \in \mathbb{R}^N} P_a(x) \quad \text{s.t.} \quad Ax = y. \\ \Leftrightarrow \min_{x \in \mathbb{R}^N} \frac{a+1}{a} \|x\|_1 - \varphi_a(x) \quad \text{s.t.} \quad Ax = y. \end{aligned}$$

We can rewrite the above optimization as

$$\min_{x \in \mathbb{R}^N} \frac{a+1}{a} \|x\|_1 + \chi(x)_{\{Ax=y\}} - \varphi_a(x) = g(x) - h(x), \quad (3.7)$$

where $g(x) = \frac{a+1}{a} \|x\|_1 + \chi(x)_{\{Ax=y\}}$ is a polyhedral convex function [27].

Choose vector $z = \partial \varphi_a(x)$, then the convex sub-problem is:

$$\min_{x \in \mathbb{R}^N} \frac{a+1}{a} \|x\|_1 - \langle z, x \rangle \quad \text{s.t.} \quad Ax = y. \quad (3.8)$$

To solve (3.8), we introduce two Lagrange multipliers u, v and define an augmented Lagrangian:

$$L_\delta(x, w, u, v) = \frac{a+1}{a} \|w\|_1 - z^t x + u^t(x - w) + v^t(Ax - y) + \frac{\delta}{2} \|x - w\|^2 + \frac{\delta}{2} \|Ax - y\|^2,$$

where $\delta > 0$. ADMM finds a saddle point (x^*, w^*, u^*, v^*) , such that:

$$L_\delta(x^*, w^*, u, v) \leq L_\delta(x^*, w^*, u^*, v^*) \leq L_\delta(x, w, u^*, v^*) \quad \forall x, w, u, v$$

by alternately minimizing L_δ with respect to x , minimizing with respect to y and updating the dual variables u and v . The saddle point x^* will be a solution to (3.8). The overall algorithm for solving the constrained TL1 is described in Algorithm (3).

Algorithm 3: DCA method for constrained TL1 minimization

Define $\epsilon_{outer} > 0$, $\epsilon_{inner} > 0$. Initialize $x^0 = 0$ and outer loop index $n = 0$

while $\|x^n - x^{n+1}\| \geq \epsilon_{outer}$ **do**

$z = \partial \varphi_a(x^n)$

Initialization of inner loop: $x_{in}^0 = w^0 = x^n$, $v^0 = 0$ and $u^0 = 0$.

Set inner index $j = 0$.

while $\|x_{in}^j - x^{j+1}\| \geq \epsilon_{inner}$ **do**

$$x_{in}^{j+1} := (A^t A + I)^{-1} (w^j + A^t y + \frac{z - u^j - A^t v^j}{\delta})$$

$$w^j = \text{shrink} \left(x_{in}^{j+1} + \frac{u^j}{\delta}, \frac{a+1}{a\delta} \right)$$

$$u^{j+1} := u^j + \delta(x_{in}^{j+1} - w^j)$$

$$v^{j+1} := v^j + \delta(Ax_{in}^{j+1} - y)$$

end while

$x^n = x_{in}^j$ and $n = n + 1$.

end while

According to DC decomposition scheme (3.7), Algorithm 3 is a polyhedral DC program. Similar convergence theorem as the unconstrained model in last section can be proved. Furthermore, due to property of polyhedral DC programs, this constrained DCA also has a finite convergence. It means that if the inner subproblem (3.8) is exactly solved, $\{x^n\}$, the sequence generated by this iterative DC algorithm, has finite subsequential limit points [27].

IV. NUMERICAL RESULTS

In this section, we use two classes of randomly generated matrices to illustrate the effectiveness of our Algorithms: DCATL1 (difference convex algorithm for transformed l_1 penalty) and its constrained version. We compare them separately with several state-of-the-art solvers on recovering sparse vectors:

- unconstrained algorithms:
 - (i) Reweighted $l_{1/2}$ [15];
 - (ii) DCA l_{1-2} algorithm [35], [19];
 - (iii) CEL0 [29]
- constrained algorithms:
 - (i) Bregman algorithm [36];
 - (ii) Yall1;
 - (iii) $L_p - RLS$ [8].

All our tests were performed on a *Lenovo* desktop with 16 GB of RAM and Intel Core processor *i7-4770* with CPU at $3.40GHz \times 8$ under 64-bit Ubuntu system.

The two classes of random matrices are:

- 1) Gaussian matrix.
- 2) Over-sampled DCT with factor F .

We did not use prior information of the true sparsity of the original signal x^* . Also, for all the tests, the computation is initialized with zero vectors. In fact, the DCATL1 does not guarantee a global minimum in general, due to nonconvexity of the problem. Indeed we observe that DCATL1 with random starts often gets stuck at local minima especially when the matrix A is ill-conditioned (e.g. A has a large condition number or is highly coherent). In the numerical experiments, by setting $x_0 = 0$, we find that DCATL1 usually produces a global minimizer. The intuition behind our choice is that by using zero vector as initial guess, the first step of our algorithm reduces to solving an unconstrained weighted l_1 problem. So basically we are minimizing TL1 on the basis of l_1 , which possibly explains why minimization of TL1 initialized by $x_0 = 0$ always outperforms l_1 .

A. Choice of Parameter: ‘ a ’

In DCATL1, parameter a is also very important. When a tends to zero, the penalty function approaches the l_0 norm. If a goes to $+\infty$, objective function will be more convex and act like the l_1 optimization. So choosing a better a will improve the effectiveness and success rate for our algorithm.

We tested DCATL1 on recovering sparse vectors with different parameter a , varying among $\{0.1 \ 0.3 \ 1 \ 2 \ 10\}$. In this test, A is a 64×256 random matrix generated by normal Gaussian distribution. The true vector x^* is also a randomly generated sparse vector with sparsity k in the set $\{8 \ 10 \ 12 \ \dots \ 32\}$. Here the regularization parameter λ was set to be 10^{-5} for all tests. Although the best λ should be dependent in general, we considered the noiseless case and $\lambda = 10^{-5}$ is small enough to approximately enforce $Ax = Ax^*$. For each a , we sampled 100 times with different A and x^* . The recovered vector x_r is accepted and recorded as one success if the relative error: $\frac{\|x_r - x^*\|_2}{\|x^*\|_2} \leq 10^{-3}$.

Fig. 2 shows the success rate using DCATL1 over 100 independent trials for various parameter a and sparsity k . From the figure, we see that DCATL1 with $a = 1$ is the best among all tested values. Also numerical results for $a = 0.3$ and $a = 2$ (near 1), are better than those with 0.1 and 10. This is because

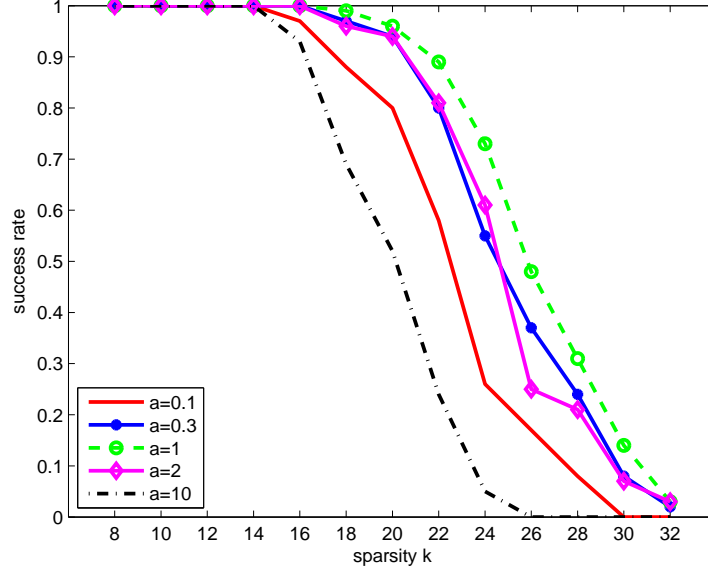


Fig. 2: Numerical tests on parameter a with $M = 64$, $N = 256$ by the unconstrained DCATL1 method.

the objective function is more non-convex at a smaller a and thus more difficult to solve. On the other hand, iterations are more likely to stop at a local ℓ_1 minima far from ℓ_0 solution if a is too large. Thus in all the following tests, we set the parameter $a = 1$.

B. Numerical Experiment for Unconstrained Algorithm

1) *Gaussian matrix*: We use $\mathcal{N}(0, \Sigma)$, the multi-variable normal distribution to generate Gaussian matrix A . Here covariance matrix is $\Sigma = \{(1 - r) * \chi_{(i=j)} + r\}_{i,j}$, where the value of ‘ r ’ varies from 0 to 0.8. In theory, the larger the r is, the more difficult it is to recover true sparse vector. For matrix A , the row number and column number are set to be $M = 64$ and $N = 1024$. The sparsity k varies among $\{5 \ 7 \ 9 \dots 25\}$.

We compare four algorithms in terms of success rate. Denote x_r as a reconstructed solution by a certain algorithm. We consider one algorithm to be successful, if the relative error of x_r to the truth solution x is less than 0.001, i.e., $\frac{\|x_r - x\|}{\|x\|} < 1.e - 3$. In order to improve success rates for all compared algorithms, we set tolerance parameter to be smaller or maximum cycle number to be higher inside each algorithm. As a result, it takes a long time to run one realization using all algorithms separately.

The success rate of each algorithm is plotted in Figure 3 with parameter r from the set: $\{0 \ 0.2 \ 0.6 \ 0.8\}$. For all cases, DCATL1 and reweighted $l_{1/2}$ algorithms (IRucLq-v) performed almost the same and both were much better than the other two, while the CEL0 has the lowest success rate.

2) *Over-sampled DCT*: The over-sampled DCT matrices A [13] [19] are:

$$A = [a_1, \dots, a_N] \in \mathbb{R}^{M \times N},$$

$$\text{where } a_j = \frac{1}{\sqrt{M}} \cos\left(\frac{2\pi\omega(j-1)}{F}\right), \quad j = 1, \dots, N, \quad (4.1)$$

and ω is a random vector, drawn uniformly from $(0, 1)^M$.

Such matrices appear as the real part of the complex discrete Fourier matrices in spectral estimation [13]. An important property is their high coherence: for a 100×1000 matrix with $F = 10$, the coherence is 0.9981, while the coherence of the same size matrix with $F = 20$, is typically 0.9999.

The sparse recovery under such matrices is possible only if the non-zero elements of solution x are sufficiently separated. This phenomenon is characterized as *minimum separation* in [5], and this

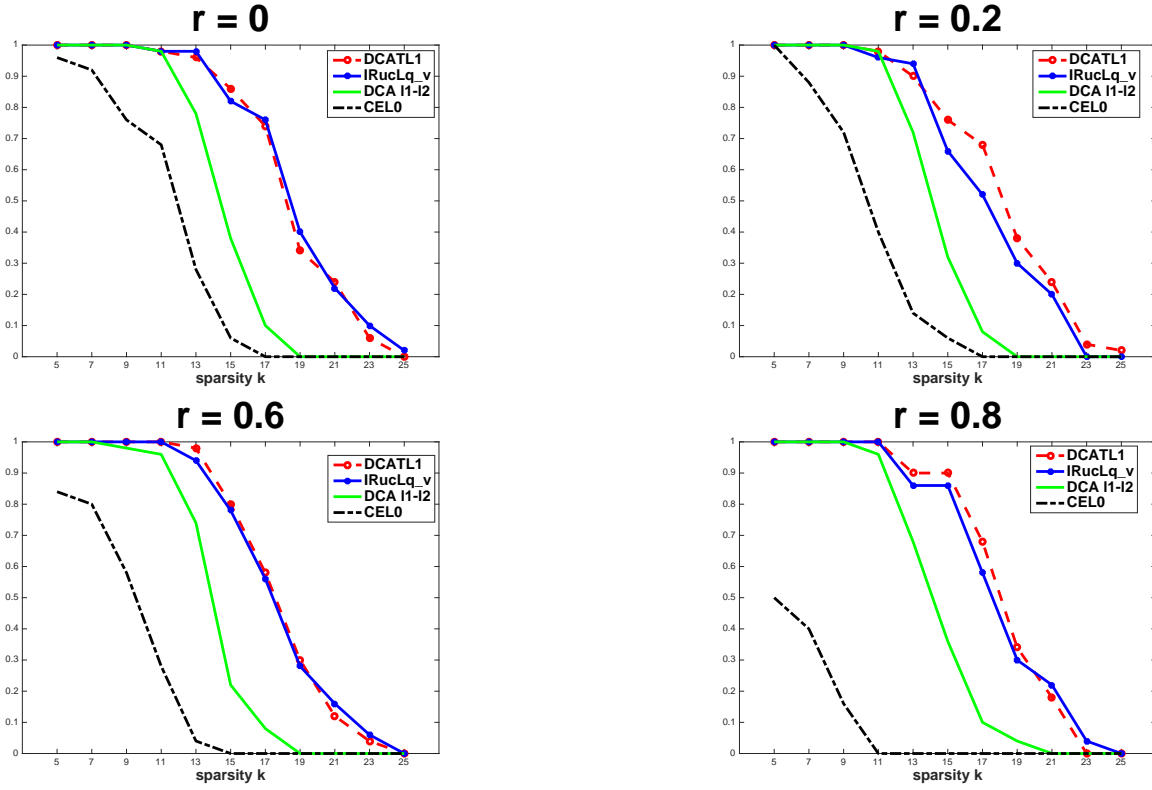


Fig. 3: Numerical tests for unconstrained algorithms under Gaussian generated matrices: $M = 64$, $N = 1024$ with different coherence r .

minimum length is referred as the Rayleigh length (RL). The value of RL for matrix A is equal to the factor F . It is closely related to the coherence in the sense that larger F corresponds to larger coherence of a matrix. We find empirically that at least $2RL$ is necessary to ensure optimal sparse recovery with spikes further apart for more coherent matrices.

Under the assumption of sparse signal with $2RL$ separated spikes, we compare those four algorithms in terms of success rate. Denote x_r as a reconstructed solution by a certain algorithm. We consider one algorithm successful, if the relative error of x_r to the truth solution x is less than 0.001, i.e., $\frac{\|x_r - x\|}{\|x\|} < 0.001$. The success rate is averaged over 50 random realizations.

Fig. 4 shows success rates for those algorithms with increasing factor F from 2 to 20. The sensing matrix is of size 100×1500 . It is interesting to see that along with the increasing of value F , DCA of $l_1 - l_2$ algorithm performs better and better, especially after $F \geq 10$, and it has the highest success rate among all. Meanwhile, reweighted $l_{1/2}$ is better for low coherent matrices. When $F \geq 10$, it is almost impossible for it to recover sparse solution for the high coherent matrix. Our DCATL1, however, is more robust and consistently performed near the top, sometimes even the best. So it is a valuable choice for solving sparse optimization problems where coherence of sensing matrix is unknown.

We further look at the success rates of DCATL1 with different combinations of sparsity and separation lengths for the over-sampled DCT matrix A . The rates are recorded in Table I, which shows that when the separation is above with the minimum length, the sparsity relative to M plays more important role in determining the success rates of recovery.

TABLE I: The success rates (%) of DCATL1 for different combination of sparsity and minimum separation lengths.

sparsity	5	8	11	14	17	20
1RL	100	100	95	70	22	0
2RL	100	100	98	74	19	5
3RL	100	100	97	71	19	3
4RL	100	100	100	71	20	1
5RL	100	100	96	70	28	1

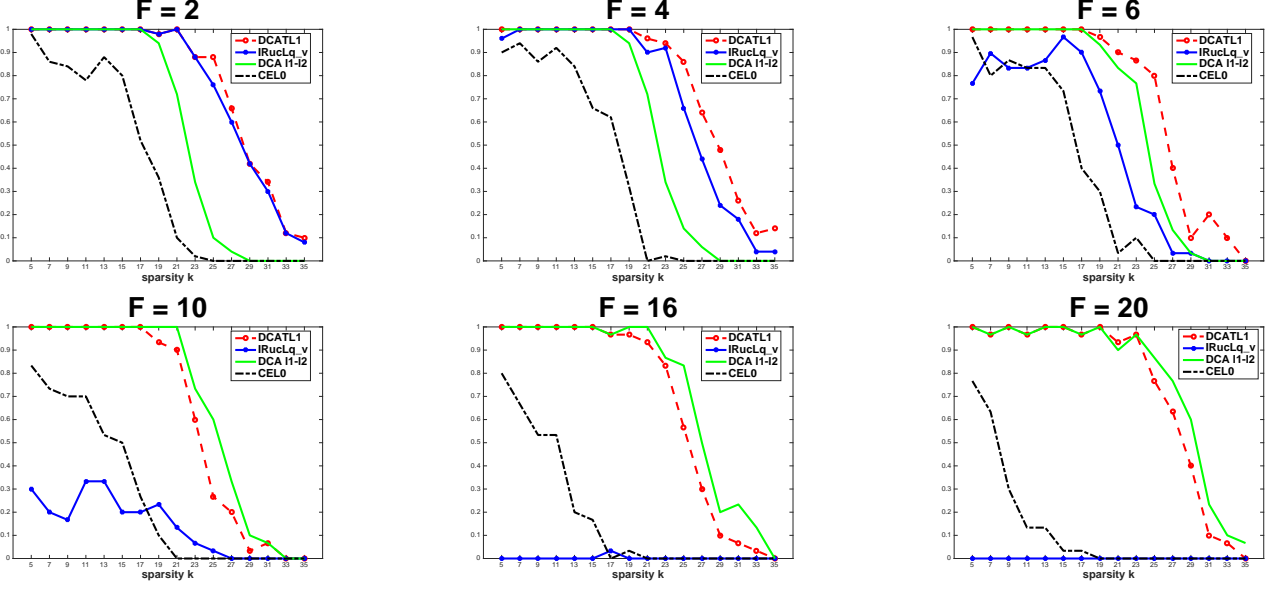


Fig. 4: Numerical test for unconstrained algorithms under over-sampled DCT matrices: $M = 100$, $N = 1500$ with different F , and peaks of solutions separated by $2RL = 2F$.

C. Numerical Experiment for Constrained Algorithm

For constrained algorithms, we performed similar numerical experiments. An algorithm is considered successful if the relative error of the numerical result x_r from the ground truth x is less than 0.001, or $\frac{\|x_r - x\|}{\|x\|} < 0.001$. We did 50 trials to compute average success rates for all the numerical experiments as for the unconstrained algorithms.

1) *Gaussian Random Matrices:* We fix parameters $(M, N) = (64, 1024)$, while covariance parameter r is varied from 0 to 0.8. Comparison is with the reweighted $l_{1/2}$ and two l_1 algorithms (Bregman and yall1). In Fig. (5), we see that $Lp - RLS$ is the best among the four algorithms with DCATL1 trailing not much behind.

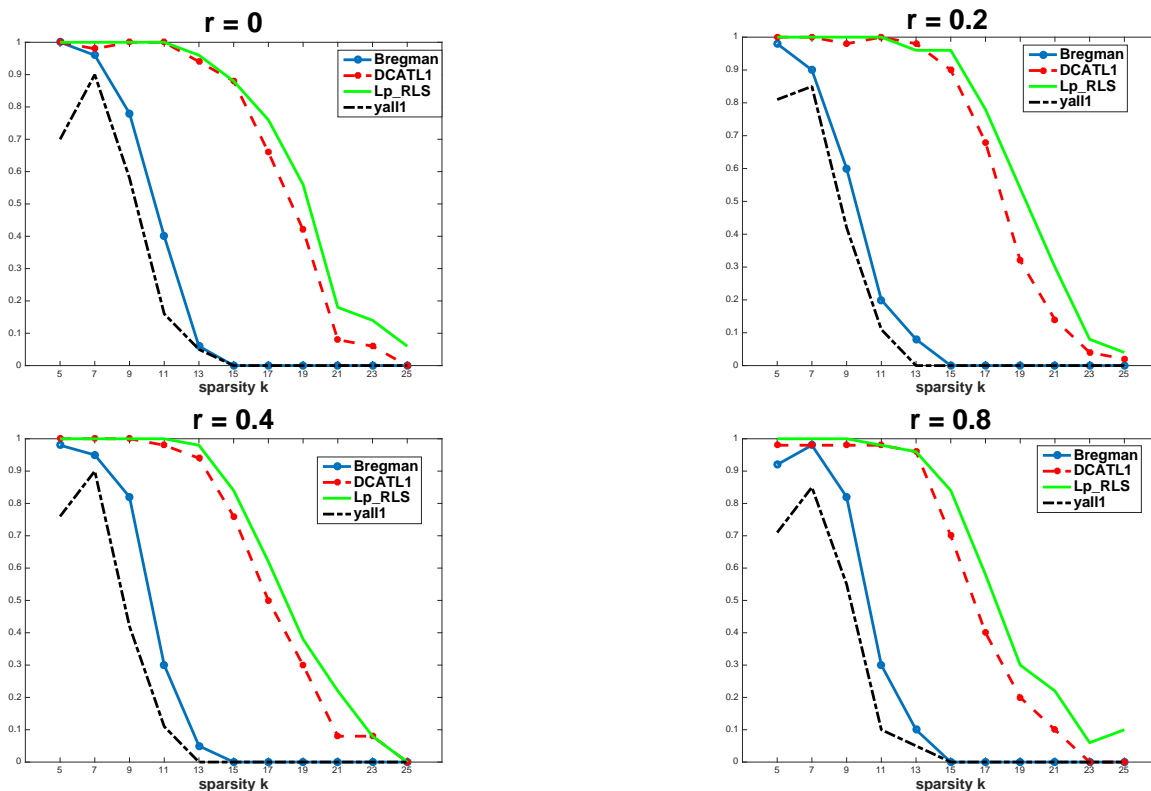


Fig. 5: Comparison of constrained algorithms for 64×1024 Gaussian random matrices with different coherence parameter r . The data points are averaged over 50 trials.

2) *Over-sampled DCT*: We fix $(M, N) = (100, 1500)$, and vary parameter F from 2 to 20, so the coherence of these matrices has a wider range and almost reaches 1 at the high end. In Fig. (6), when F is small, say $F = 2, 4$, $Lp-RLS$ still performs the best, similar to the case of Gaussian matrices. However, with increasing F , the success rates for $Lp-RLS$ decline quickly, worse than the Bregman l_1 algorithm at $F = 6, 10$. The performance for DCATL1 is very stable and maintains a high level consistently even at the very high end of coherence ($F = 20$).

V. CONCLUDING REMARKS

We have studied compressed sensing problem with the transformed l_1 penalty function for both the unconstrained and constrained models. We presented a theory on the uniqueness and l_0 equivalence of the global minimizer of the unconstrained model under RIP and analyzed properties of local minimizers. We showed two DC algorithms along with a convergence theory.

In numerical experiments, DCATL1 is on par with the best method reweighted $l_{1/2}$ ($Lp-RLS$) in the unconstrained (constrained) model, using incoherent Gaussian matrices A . For highly coherent over-sampled DCT matrices, DCATL1 is comparable to the best method DCA l_1-l_2 algorithm. For random matrices of varying degree of coherence, we tested Gaussian and over-sampled DCT sensing matrices. The DCATL1 algorithm is the most robust for constrained and unconstrained models alike.

In future work, we plan to develop TL1 algorithms for imaging processing applications such as deconvolution and deblurring.

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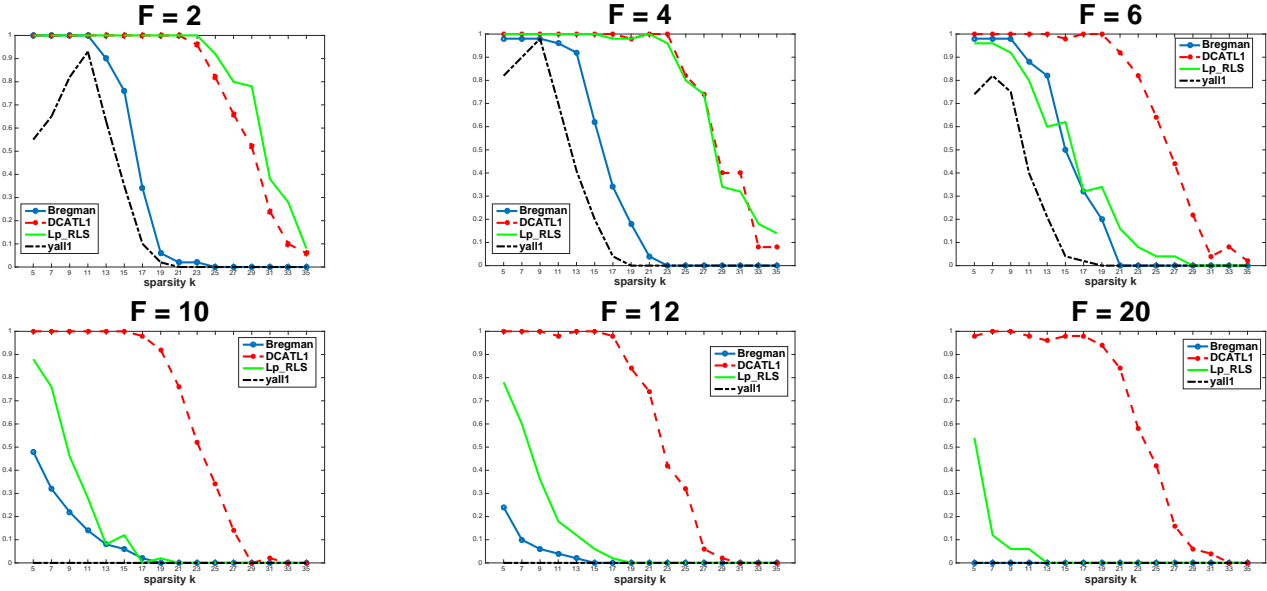


Fig. 6: Comparison of success rates of constrained algorithms for the over-sampled DCT random matrices: $(M, N) = (100, 1500)$ with different F values, peak separation by $2RL = 2F$.

APPENDIX PROOF OF TL1 EXACT SPARSE RECOVERY THEOREM

Proof: The proof generally follows the lines of arguments in [2] and [7], while using special properties of the penalty function ρ_a .

For simplicity, we denote β_C by β and β_C^0 by β^0 .

Define the function:

$$f(a) = \frac{a^2}{(a+1)^2} \frac{R}{|T|} (1 - \delta_{R+|T|}) - 1 - \delta_R$$

It is continuous and increasing in the parameter a . Note that at $a = 0$, $f(0) = -1 - \delta_M < 0$, and as $a \rightarrow \infty$, $f(a) \rightarrow \frac{R}{|T|} (1 - \delta_{R+|T|}) - 1 - \delta_R > 0$ by (2.12). There exists a constant a^* , such that $f(a^*) = 0$. The number a^* depends on the RIP of matrix A only, and so it is independent of the scalar C .

For $a > a^*$:

$$\delta_R + \frac{a^2}{(a+1)^2} \frac{R}{|T|} \delta_{R+|T|} < \frac{a^2}{(a+1)^2} \frac{R}{|T|} - 1. \quad (1.2)$$

Let $e = \beta - \beta^0$, and we want to prove that the vector $e = 0$. It is clear that, $e_{T^c} = \beta_{T^c}$, since T is the support set of β^0 . By the triangular inequality of ρ_a , we have:

$$P_a(\beta^0) - P_a(e_T) = P_a(\beta^0) - P_a(-e_T) \leq P_a(\beta_T).$$

Then

$$\begin{aligned} P_a(\beta^0) - P_a(e_T) + P_a(e_{T^c}) &\leq P_a(\beta_T) + P_a(\beta_{T^c}) \\ &= P_a(\beta) \\ &\leq P_a(\beta^0) \end{aligned}$$

It follows that:

$$P_a(\beta_{T^c}) = P_a(e_{T^c}) \leq P_a(e_T). \quad (1.3)$$

Now let us arrange the components at T^c in the order of decreasing magnitude of $|e|$ and partition into L parts: $T^c = T_1 \cup T_2 \cup \dots \cup T_L$, where each T_j has R elements (except possibly T_L with less). Also

denote $T = T_0$ and $T_{01} = T \cup T_1$. Since $Ae = A(\beta - \beta^0) = 0$, it follows that

$$\begin{aligned}
0 &= \|Ae\|_2 \\
&= \|A_{T_{01}}e_{T_{01}} + \sum_{j=2}^L A_{T_j}e_{T_j}\|_2 \\
&\geq \|A_{T_{01}}e_{T_{01}}\|_2 - \sum_{j=2}^L \|A_{T_j}e_{T_j}\|_2 \\
&\geq \sqrt{1 - \delta_{|T|+R}} \|e_{T_{01}}\|_2 - \sqrt{1 + \delta_R} \sum_{j=2}^L \|e_{T_j}\|_2
\end{aligned} \tag{1.4}$$

At the next step, we derive two inequalities between the l_2 norm and function P_a , in order to use the inequality (1.3). Since

$$\begin{aligned}
\rho_a(|t|) &= \frac{(a+1)|t|}{a+|t|} \leq \left(\frac{a+1}{a}\right)|t| \\
&= \left(1 + \frac{1}{a}\right)|t|
\end{aligned}$$

we have:

$$\begin{aligned}
P_a(e_{T_0}) &= \sum_{i \in T_0} \rho_a(|e_i|) \\
&\leq \left(1 + \frac{1}{a}\right) \|e_{T_0}\|_1 \\
&\leq \left(1 + \frac{1}{a}\right) \sqrt{|T|} \|e_{T_0}\|_2 \\
&\leq \left(1 + \frac{1}{a}\right) \sqrt{|T|} \|e_{T_{01}}\|_2.
\end{aligned} \tag{1.5}$$

Now we estimate the l_2 norm of e_{T_j} from above in terms of P_a . It follows from β being the minimizer of the problem (2.11) and the definition of x_C (2.8) that

$$P_a(\beta_{T^c}) \leq P_a(\beta) \leq P_a(x_C) \leq 1.$$

For each $i \in T^c$, $\rho_a(\beta_i) \leq P_a(\beta_{T^c}) \leq 1$. Also since

$$\begin{aligned}
\frac{(a+1)|\beta_i|}{a+|\beta_i|} &\leq 1 \\
\Leftrightarrow (a+1)|\beta_i| &\leq a+|\beta_i| \\
\Leftrightarrow |\beta_i| &\leq 1
\end{aligned} \tag{1.6}$$

we have

$$|e_i| = |\beta_i| \leq \frac{(a+1)|\beta_i|}{a+|\beta_i|} = \rho_a(|\beta_i|) \quad \text{for every } i \in T^c.$$

It is known that function $\rho_a(t)$ is increasing for non-negative variable $t \geq 0$, and

$$|e_i| \leq |e_k| \quad \text{for } \forall i \in T_j \quad \text{and } \forall k \in T_{j-1}$$

,where $j = 2, 3, \dots, L$. Thus we will have

$$\begin{aligned}
|e_i| &\leq \rho_a(|e_i|) \leq P_a(e_{T_{j-1}})/R \\
\Rightarrow \|e_{T_j}\|_2^2 &\leq \frac{P_a(e_{T_{j-1}})^2}{R} \\
\Rightarrow \|e_{T_j}\|_2 &\leq \frac{P_a(e_{T_{j-1}})}{R^{1/2}} \\
\Rightarrow \sum_{j=2}^L \|e_{T_j}\|_2 &\leq \sum_{j=1}^L \frac{P_a(e_{T_j})}{R^{1/2}}
\end{aligned} \tag{1.7}$$

Finally, plug (1.5) and (1.7) into inequality (1.4) to get:

$$\begin{aligned}
0 &\geq \sqrt{1 - \delta_{|T|+R}} \frac{a}{(a+1)|T|^{1/2}} P_a(e_T) - \sqrt{1 + \delta_R} \frac{1}{R^{1/2}} P_a(e_T) \\
&\geq \frac{P_a(e_T)}{R^{1/2}} \left(\sqrt{1 - \delta_{R+|T|}} \frac{a}{a+1} \sqrt{\frac{R}{|T|}} - \sqrt{1 + \delta_R} \right)
\end{aligned} \tag{1.8}$$

By (1.2), the factor $\sqrt{1 - \delta_{R+|T|}} \frac{a}{a+1} \sqrt{\frac{R}{|T|}} - \sqrt{1 + \delta_R}$ is strictly positive, hence $P_a(e_T) = 0$, and $e_T = 0$. Also by inequality (1.3), $e_{T^c} = 0$. We have proved that $\beta_C = \beta_C^0$. The equivalence of (2.11) and (2.10) holds. ■

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